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YIELD CRITERION FOR AN ORTHOTROPICALLY REINFORCED SLAB

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Abstract—This paper deals with the problem of formulating a yield criterion for an orthotropically reinforced slab. The case of an initially orthotropic slab subject to bending is taken into account and any cross-section of it is assumed to have rigid-plastic properties. The possibility of the kinking of reinforcement bars on the yield hinge is considered and the so-called partial kinking yield criterion is formulated in such a way that the vanishing value of the twisting moment on the fracture line can be observed for any inclination of it with respect to the directions of reinforcement.

The proposed criterion is dependent upon a single experimental coefficient which is comparatively easy to determine and the limiting values of it are theoretically assessed. The upper limiting case is called the complete kinking theory, the lower one yields the minimum kinking theory. The commonly used 'rectangular' yield criterion which proves kinematically impermissible and which cannot prevent the plastic hinge from having non-zero value of torsion gives even lower values of yield moment than minimum kinking theory and thus appears far too pessimistic.

THE DETERMINATION OF ULTIMATE BENDING MOMENT ON THE FRACTURE LINE

IN THE limit analysis of the bending of an orthotropically reinforced rigid-plastic slab the 'rectangular' yield criterion is at the time being in common use. There are two main reasons why this criterion is so often applied: relative simplicity of its analytical formulation and its conservativeness in determination of the ultimate moment values. However, this criterion is not free from criticism [1] and one of the fundamental defects of it is that the implicit assumption of a 'stepped' fracture line leads to kinematically impermissible mechanism of rotation of the adjacent rigid parts of a slab. As a consequence of this, the twist cannot vanish on the yield hinge except in a trivial case when the fracture line is perpendicular to the initial direction of the reinforcement. Thus the ultimate moment on the plastic hinge is not a principal moment and therefore leads to an ambiguous situation which contradicts the construction of yield condition (yield surface) taking into account the statically admissible field of moments m_x, m_y, m_{xy} acting on a slab.

The need of a yield criterion for an orthotropic slab which disposes of this discrepancy and renders the problem of formulating of the yield condition more consistent seems to be very urgent.

Let us consider the result of an introductory analysis worked out in paper [2] and suitable for the case of initial orthotropy. Taking into account a kinematically permissible mechanism of the fracture line generation (shown in Fig. 1) in which the kinking of

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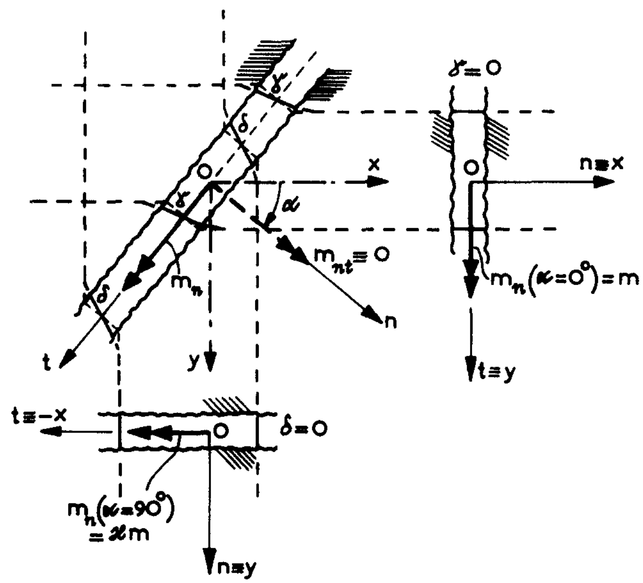

FIG. 1. Location of bars and yield-moment vectors for the partial kinking theory for orthotropically reinforced slab, $\kappa < 1$.

reinforcement bars due to crushing of concrete in the neighbourhood of the kinks is allowed we have obtained

$$\frac{m_n}{m} = \cos\alpha\cos\gamma + \kappa\sin\alpha\cos\delta \tag{1}$$

$$\frac{m_{nt}}{m} = \cos\alpha\sin\gamma - \kappa\sin\alpha\sin\delta \tag{2}$$

where

m_n — ultimate (plastic or yield) bending moment on the fracture line, normal n of which is inclined at the angle α, $(0 \le \alpha \le 90°)$, to the direction of 'stronger' reinforcement (x-axis),

m_{nt} — twisting moment on the fracture line,

$m = m_n\,(\alpha = 0°)$,

$\kappa = \dfrac{m_n\,(\alpha = 90°)}{m} \le 1$—coefficient of initial orthotropy,

γ, δ — kinking angles of reinforcement bars measured with respect to the normal n.

THE INTERDEPENDENCE OF THE KINKING ANGLES γ AND δ

Requiring to have a physically well justified condition $m_{nt} = 0$ always (i.e. for any α) fulfilled we find immediately from (2) that the equation

$$\frac{\sin\gamma}{\sin\delta} = \kappa\,\mathrm{tg}\,\alpha \tag{3}$$

must hold. The expression (3) means that the directions of bars across the open fracture line adjust themselves in a specific way according to prescribed orthotropy and a current value of inclination α.

Next, a further relation

$$\gamma = g(\alpha) \tag{4}$$

has to be assumed. It is required that the function (4) should be smooth, ascending, and satisfying the condition $\gamma = 0$, at $\alpha = 0$. Two additional requirements, viz. $\gamma = \delta$, at $\kappa = 1$, $\alpha = 45°$, and $\delta = 0$, at $\alpha = 90°$ are thanks to relation (3) automatically fulfilled. The function (4) in the form

$$\sin\gamma = A\sin\alpha \tag{5}$$

where

$$A = A(\kappa, \mu) \tag{6}$$

appears, for instance, promising.

The parameter μ which governs in fact the relation (4) is defined by

$$\mu = \frac{m_n(\bar{\alpha})}{m} \tag{7}$$

where $\bar{\alpha}$ stand for some specific but virtually arbitrary value of $0° < \alpha < 90°$. The parameter μ has to be investigated by means of an experiment which is clearly described by (7) and is fully responsible for the now 'unrectangularness' of the known conservative yield condition that is usually expressed as

$$\frac{m_n}{m} = \cos^2\alpha + \kappa\sin^2\alpha. \tag{8}$$

THE FINAL FORMULATION OF THE PARTIAL KINKING YIELD CRITERION

Remembering (3) and (5) it is readily obtained that

$$\sin\delta = \frac{A}{\kappa}\cos\alpha \tag{9}$$

hence

$$\cos\gamma = \sqrt{(1 - A^2\sin^2\alpha)} \tag{10}$$

$$\cos\delta = \sqrt{\left(1 - \frac{A^2}{\kappa^2}\cos^2\alpha\right)}. \tag{11}$$

Eventually the partial kinking yield criterion (1) takes the form

$$\frac{m_n}{m} = \cos\alpha\sqrt{(1 - A^2\sin^2\alpha)} + \kappa\sin\alpha\sqrt{\left(1 - \frac{A^2}{\kappa^2}\cos^2\alpha\right)} \tag{12}$$

$$m_{nt} \equiv 0.$$

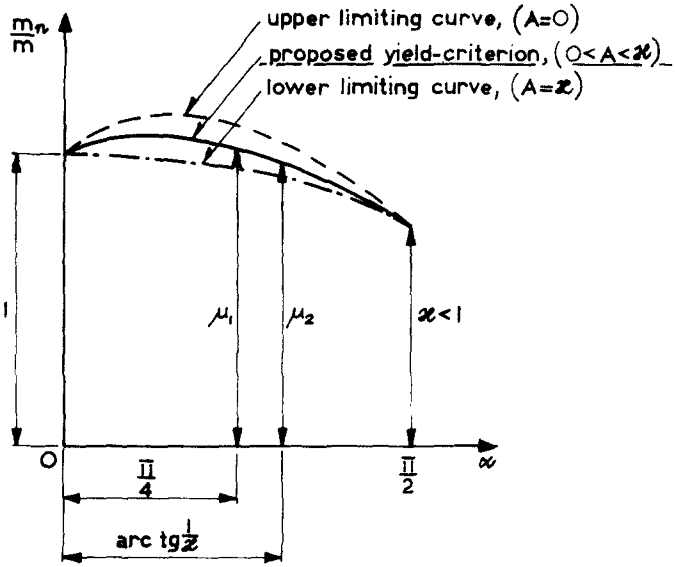

FIG. 2. Proposed yield-criterion and a governing coefficient μ_1 (or μ_2) for orthotropically reinforced slab.

Now, the two convenient ways of fixing the parameter μ appear:

Case 1. $\bar{\alpha} = 45°$ (see Fig. 2) irrespective of the value of κ, hence

$$\mu_1 = \frac{m_n(\bar{\alpha} = 45°)}{m}. \tag{13}$$

Having (10) and (11) we easily find

$$\cos\bar{\gamma} = \sqrt{\left(1-\frac{A_1^2}{2}\right)}, \qquad \cos\bar{\delta} = \sqrt{\left(1-\frac{A_1^2}{2\kappa^2}\right)},$$

and remembering (1) and (13) get A, introduced in (5) and defined by (6)

$$A_1 = \sqrt{\left[2-\frac{1}{4\mu_1^2}(2\mu_1^2-\kappa^2+1)^2\right]}. \tag{14}$$

From the condition $A_1 \geq 0$ the following inequalities are obtained:

$$\frac{1-\kappa}{\sqrt{2}} \leq \mu_1 \leq \frac{1+\kappa}{\sqrt{2}}.$$

Another necessary condition which binds the value of μ_1 is $A_1 \leq \kappa$, otherwise the relation (9) could not hold. Solution of this inequality yields

$$\mu_1 \leq \tfrac{1}{2}[\sqrt{(2-\kappa^2)}-\kappa], \qquad \tfrac{1}{2}[\sqrt{(2-\kappa^2)}+\kappa] \leq \mu_1.$$

The joint condition

$$\kappa \geq A_1 \geq 0 \tag{15}$$

is fulfilled when

$$\tfrac{1}{2}[\sqrt{(2-\kappa^2)}+\kappa] \leq \mu_1 \leq \frac{1+\kappa}{\sqrt{2}}. \tag{16}$$

This prescribes the interval of values μ_1 which are expected to be obtained from the experiment.

It is easy to observe that the right-hand side equality of (15) and (16) is appropriate for the complete kinking theory ($\gamma = \delta = 0$) whereas the left-hand side equality of (15) and (16) fits to the case which can be called the minimum kinking theory and will be discussed in detail later on.

Case 2. We fix such an angle $\bar{\alpha}$ that results in the equality of both kinking angles, $\bar{\gamma} = \bar{\delta}$. Having in mind expression (3) it is readily found that

$$\operatorname{tg} \bar{\alpha} = \frac{1}{\kappa} \tag{17}$$

hence in this case we define μ as

$$\mu_2 = \frac{m_n[\alpha = \operatorname{arc\,tg}(1/\kappa)]}{m}. \tag{18}$$

From (10) and (11) we find

$$\cos \bar{\gamma} = \cos \bar{\delta} = \sqrt{\left(1-\frac{A_2^2}{1+\kappa^2}\right)}$$

and bearing in mind (1) we get from (18) that

$$A_2 = \sqrt{\left[1+\kappa^2-\mu_2^2\left(\frac{1+\kappa^2}{2\kappa}\right)^2\right]}. \tag{19}$$

Again the simultaneous condition

$$\kappa \geq A_2 \geq 0 \tag{20}$$

results in

$$\frac{2\kappa}{1+\kappa^2} \leq \mu_2 \leq \frac{2\kappa}{\sqrt{(1+\kappa^2)}}. \tag{21}$$

Any experimentally obtained coefficient μ_2 ought to lie within these two limiting values.

It is again easy to show that when the right-hand side equalities of (20) and (21) hold we have complete kinking and when the left-hand side equalities are satisfied then minimum kinking occurs.

Since the problem of determining of the yield criterion within all the assumptions rendered is unique, $A_1 = A_2 = A$, hence there must exist a relation between coefficients μ_1 and μ_2 which appears from (14) and (19) to be

$$\mu_2 = \frac{2\kappa}{1+\kappa^2}\sqrt{\left[\left(\frac{2\mu_1^2-\kappa^2+1}{2\mu_1}\right)^2+\kappa^2-1\right]}. \tag{22}$$

At this stage neither of those two ways of defining μ appears preferential and both of them serve the same purpose, i.e. to assess the degree of 'unrectangularness' of the yield criterion on the basis of tests. Both of these seem fairly convenient and it is worth noticing that in the case of isotropy, $\kappa = 1$, both yield the same definition.

UPPER LIMITING CASE—THE COMPLETE KINKING YIELD CRITERION

The case $A = 0$ ($\gamma = \delta = 0$) results, as has been pointed out previously, in the complete kinking yield criterion

$$\left.\begin{aligned} \frac{m_n}{m} &= \cos\alpha + \kappa\sin\alpha \\ m_{nt} &\equiv 0 \end{aligned}\right\}. \tag{23}$$

The values of ultimate moment derived from this criterion are the most optimistic ones. From the kinematical point of view it means that all the bars across the fracture line remain parallel to the normal n (i.e. perpendicular to the fracture line itself).

The criterion (23) does not depend upon any experimental coefficient μ but if the complete kinking theory were true the coefficient μ taken from the test should yield the values

$$\mu_1 = \frac{1+\kappa}{\sqrt{2}} \qquad \text{or} \qquad \mu_2 = \frac{2\kappa}{\sqrt{(1+\kappa^2)}}.$$

LOWER LIMITING CASE—THE MINIMUM KINKING YIELD CRITERION

The case $A = \kappa$, ($\sin\gamma = \kappa\sin\alpha$, $\sin\delta = \cos\alpha$) results in the minimum kinking criterion

$$\left.\begin{aligned} \frac{m_n}{m} &= \cos\alpha\sqrt{(1-\kappa^2\sin^2\alpha)} + \kappa\sin^2\alpha \\ m_{nt} &\equiv 0 \end{aligned}\right\}. \tag{24}$$

The values of plastic moment calculated from this criterion are the most pessimistic ones (provided $m_{nt} = 0$ holds) and the reinforcement bars adjust themselves in such a way that

$$\left.\begin{aligned} \gamma &= \text{arc}\sin(\kappa\sin\alpha) \\ \delta &= 90^\circ - \alpha \end{aligned}\right\}. \tag{25}$$

It means (see Fig. 3) that kinking exists only in the 'stronger' reinforcement bars, whereas the bars in perpendicular direction remain straight (strictly speaking their projections on the x, y-plane generate straight lines). Hence the lower limiting case of the partial kinking theory, viz. the minimum kinking theory is in fact already kinematically impermissible.

The criterion (24) does not depend on any coefficient μ either but we can similarly conclude that if the minimum kinking theory had been applicable then μ taken by means

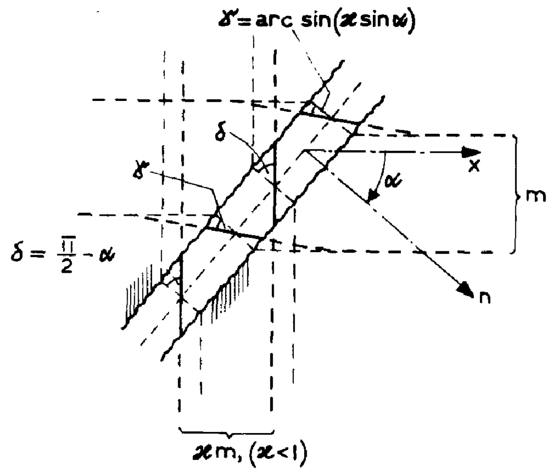

FIG. 3. Location of bars in the case of minimum kinking theory, $A = \kappa$.

of tests should have given the values

$$\mu_1 = \tfrac{1}{2}[\sqrt{(2-\kappa^2)}+\kappa] \qquad \text{or} \qquad \mu_2 = \frac{2\kappa}{1+\kappa^2}.$$

The author believes that in reality coefficient A lies somewhere between $A = 0$ and $A = \kappa$. In other words the kinking of bars appears to be 'partial'. Adequate experiments should be performed and coefficient μ (thereby A) should be found.

THE 'NO KINKING' YIELD CRITERION

The 'stepped line' yield criterion which does not permit any reinforcement bar to be kinked in the x, y-plane of a slab, ($\gamma = \alpha$, $\delta = 90° - \alpha$) is usually written down as follows

$$\left.\begin{aligned} \frac{m_n}{m} &= \cos^2\alpha + \kappa \sin^2\alpha \\ \frac{m_{nt}}{m} &= (1-\kappa)\sin\alpha\cos\alpha \end{aligned}\right\}. \tag{26}$$

It is, however, strongly stressed that this criterion cannot be derived as a special case from the partial kinking theory for any value of A, ($0 \le A \le \kappa$, $\kappa < 1$) because it cannot in fact exist for the orthotropically reinforced slab if we wish to preserve the no-twist ($m_{nt} = 0$) requirement.

Only in the case of an isotropic slab, $\kappa = 1$, which entails immediately $A = 1$ if no kinking is permitted, the condition $m_{nt} = 0$ is identically fulfilled and the stepped line yield criterion is qualitatively comparable with the more general partial kinking criterion.

In other words the minimum kinking theory plays the same role for an orthotropically reinforced slab as the stepped line (no kinking) theory does for an isotropically reinforced slab. Both of them are lower limiting cases (already kinematically impermissible) of a more realistic partial kinking theory for an orthotropic and an isotropic slab, respectively.

If the stepped line theory for an orthotropic slab had existed then the coefficient μ should have been

$$\mu_1 = \frac{1+\kappa}{2} \quad \text{or} \quad \mu_2 = \frac{\kappa(1+\kappa)}{1+\kappa^2}$$

which does not, however, seem in the light of previous discussion to be possible.

NUMERICAL EXAMPLE AND GRAPHICAL REPRESENTATION

In order to illustrate the partial kinking yield criterion obtained and visualize its limiting cases let us assume, for example,

$$\kappa = 0{\cdot}7, \qquad 0{\cdot}965 < \mu_1 = 1{\cdot}12 < 1{\cdot}202$$

$$(0{\cdot}940 < \mu_2 = 1{\cdot}07 < 1{\cdot}147, \qquad 0 < A = 0{\cdot}429 < 0{\cdot}7).$$

The results of numerical calculations and their graphical representation are shown in Figs. 4, 5 and 6. Looking at Fig. 5 a rather striking property of functions $\gamma(\alpha)$ and $\delta(\alpha)$ can be seen.

When $\alpha = 0°$, then $\gamma = 0°$ (see expression (5)), but $\delta \neq 90°$ (see expression (9)) and when $\alpha = 90°$, then $\delta = 0°$ but $\gamma \neq 90°$.

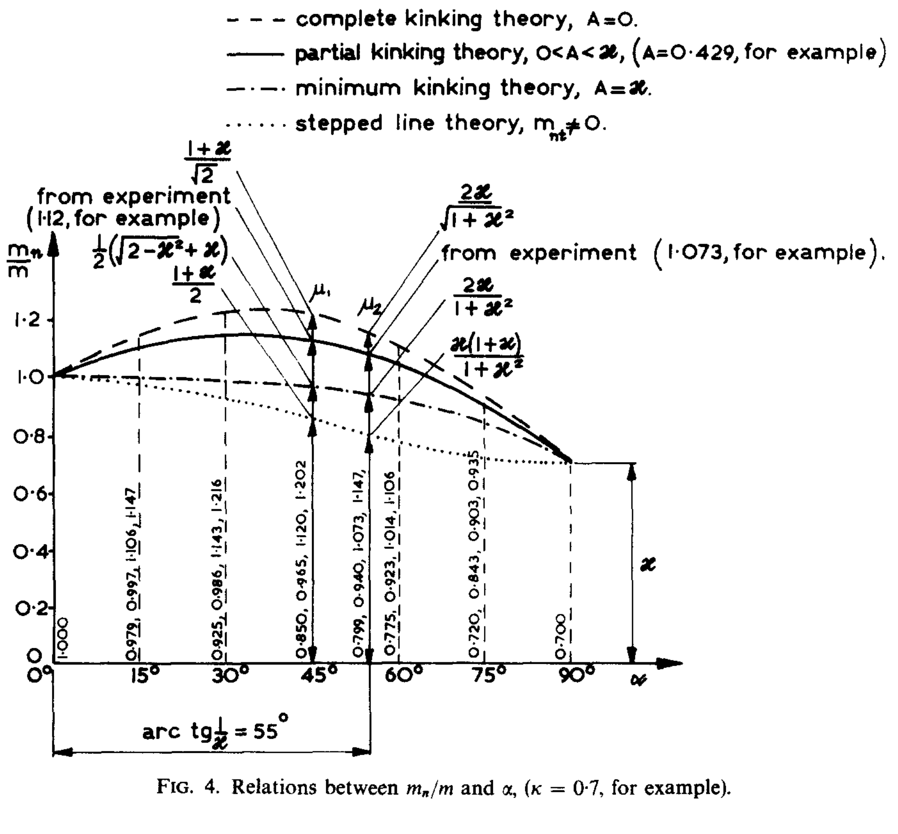

FIG. 4. Relations between m_n/m and α, ($\kappa = 0{\cdot}7$, for example).

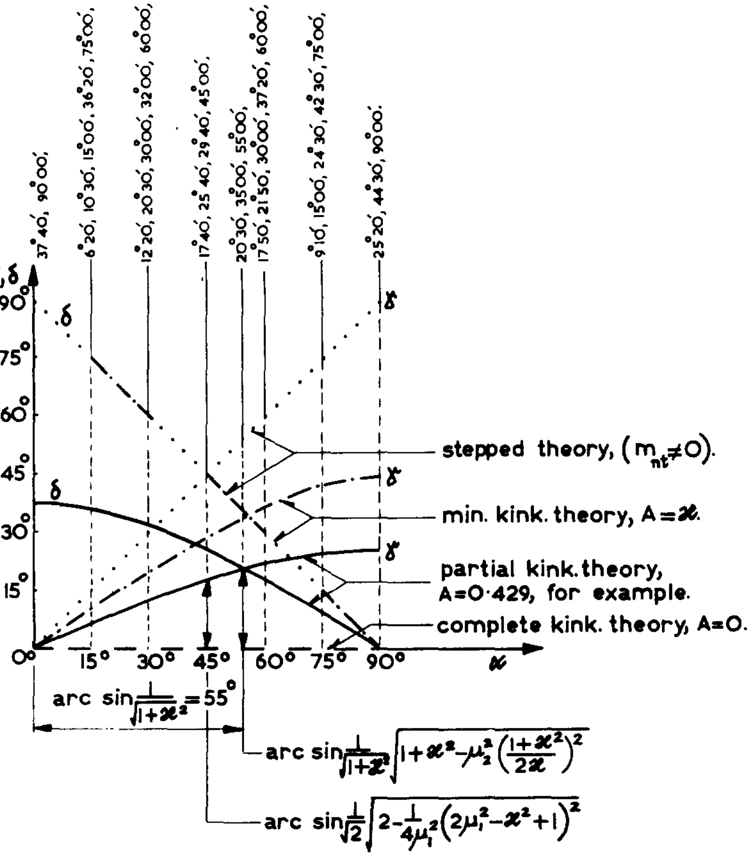

FIG. 5. Relations between γ, δ and α, (κ = 0·7, for example).

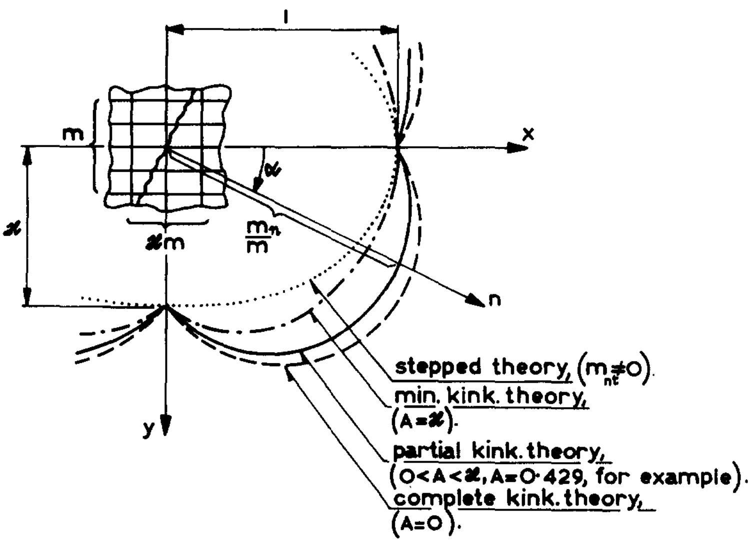

FIG. 6. Graphical representation of the proposed criterion in polar co-ordinates m_n/m, α–solid curve, (κ = 0·7, for example).

The values of those non-right angles δ and γ are as follows

$$\delta(\alpha = 0°) = \arc\sin \frac{A}{\kappa}$$

$$\gamma(\alpha = 90°) = \arc\sin A.$$

This geometrical singularity vanishes only in the trivial case of the stepped line criterion for an isotropically reinforced slab, $A = \kappa = 1$.

However, it does not affect the distribution of a function $m_n(\alpha)$ (Figs. 2 and 4) in any disturbing way, neither does it violate the condition $m_{nt} = 0$ for $\alpha = 0°$ and $\alpha = 90°$. There is only one skew bar across the open fracture line $\alpha = 0°$ or $\alpha = 90°$ and it does not produce any twist because its spacing tends to infinity.

CONCLUSIONS

The partial kinking yield criterion formulated above enables one to assess the values of ultimate plastic moments on the fracture line more realistically, thus rendering the upper bound solutions of slabs (according to the kinematic approach—yield line theory) closer to the results expected in the nature.

Assuming a concept of kinking of reinforcement bars and making the function $m_n(\alpha)$ dependent upon an experimental coefficient μ which is fairly easy to fix by means of relatively simple tests permits an assessment of yield moments less conservatively.

In order to make the lower bound solutions (according to the statical approach) possible the appropriate yield condition (yield surface) has to be constructed. The author proposes to formulate such a yield condition based on the concept of kinking thus making both approaches more consistent and comparable. However, it is supposed that the analytical form of such yield condition might be significantly more complicated than that which can be worked out on the basis of Johansen's concept of yield line (cf. [3] and [4]).

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REFERENCES

[1] R. H. WOOD, *Plastic and Elastic Design of Slabs and Plates*. Thames and Hudson, London (1961).
[2] M. W. KWIECINSKI, Yield criterion for initially isotropic reinforced slab, *Mag. Concr. Res.* Vol. 17, No. 51, June 1965.
[3] K. W. JOHANSEN. *Yield Line Theory*. Cement and Concrete Association, London (1962).
[4] M. P. NIELSEN, Limit analysis of reinforced concrete slabs, *Acta polytech, scand.* Ci 26 (1964).

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Résumé—Le problème étudié par l'auteur est d'établir un critère d'écoulement pour une poutre portant un renfort orthotropique. Il prend le cas d'une poutre, avec renforcement orthotropique, soumise à un effort de flexion et il admet que toute ses sections transversales possèdent des propriétés plastico-rigides. L'auteur étudie la possibilité de flambage des renforts à la charnière d'écoulement et il exprime ce qu'il appelle le critère d'écoulement de flambage partiel, de telle manière, que la disparition du moment de torsion à la ligne de rupture puisse être observée pour toutes ses inclinaisons par rapport à la direction du renfort. Le critère proposé par l'auteur ne dépend que d'un seul coefficient expérimental qui est relativement facile à déterminer et ses valeurs limites sont établies théoriquement. La valeur limite supérieure correspond au cas, appelé par l'auteur, théorie complète du flambage et la valeur inférieure correspond à la théorie du flambage minimum. Le critère de déformation dite — rectangulaire —, utilisé habituellement, dont la cinématique est inadmissible et qui ne peut empêcher le moment de torsion de s'annuler à la charnière plastique, donne même des valeurs plus faibles, pour le moment d'écoulement, que la théorie du flambage minimum. Il est, par conséquent, beaucoup trop défavorable.

Абстракт—Эта статья рассматривает проблему формулирования критерия податливости для ортотропически укрепленной плиты. Учитывается случай с начально ортотропической плитой, подлежащей изгибу, и полагается, что любое поперечное сечение ее имеет жестко-пластичные свойства. Рассматривается возможность перегиба стержней арматуры на пружинящем шарнире, а так называемый критерий податливости при частичном перегибе формулируется таким образом, что стремящаяся к нулю величина крутящего момента на линии излома может быть измерена при любом наклоне последией по отношению к направлению укрепления. Предлагаемый критерий зависит от единого экспериментального коэффициента, который определяется сравнительно просто и предельные величины которого оценены теоретически. Верхний предельный случай называется теорией полного перегиба, тогда как нижний дает теорию минимального перегиба. Употребляемый обычно критерий "прямоугольной" податливости, который оказывается кинематически недопустимым и который не может предохранить пластический шарнир от ненулевой величины кручения, дает еще более низкие величины момента податливости, чем теория минимального перегиба, и поэтому является слишком заниженным.